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2005 J. Phys. A: Math. Gen. 38 461

(http://iopscience.iop.org/0305-4470/38/2/012)

J. Phys. A: Math. Gen. **38** (2005) 461–483 doi:10.1088/0305-4470/38/2/012

Weyl-ordered polynomials in fractional-dimensional quantum mechanics

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Received 25 June 2004, in final form 19 October 2004
Published 15 December 2004
Online at stacks.iop.org/JPhysA/38/461

Abstract

We develop algebraic properties of Weyl-ordered polynomials in the momentum and position operators P, Q which satisfy the R-deformed Heisenberg algebra, representations of which describe quantum mechanics in fractional dimensions. By viewing Weyl-ordered polynomials as tensor operators with respect to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ we derive a specific form for these polynomials, including an expression in terms of hypergeometric functions, and determine various algebraic properties such as recurrence relations, symmetries, and also a general product formula from which all commutators and anti-commutators may be calculated. We briefly discuss several applications to quantum mechanics in fractional dimensions.

PACS numbers: 03.65.−w, 03.65.Ca, 03.65.Fd, 03.65.Ge

1. Introduction

In the quantization of classical systems one encounters an infinite number of quantum operators corresponding to any classical expression, due in part to the multitude of possible orderings of the noncommuting quantum operators. In their fundamental paper of 1925, Born and Jordan [1] chose a specific ordering of the quantum mechanical momentum and position operators P, Q, satisfying $\mathrm{i}[P, Q] = 1$, in order to define a Hermitean Hamiltonian $H(P, Q)$. In their prescription the multinomial $p^m q^n$ in the classical momentum and position coordinates p, q is replaced in the quantum theory according to

$$p^m q^n \longrightarrow \frac{1}{m+1} \sum_{r=0}^{m} P^{m-r} Q^n P^r .$$

In 1927 Weyl [2] specified another ordering by means of the Fourier transform, which also defines Hermitean operators but, in contrast to the Born–Jordan scheme, is symmetric under the interchange of P, Q. Other operator orderings have also been considered in various contexts, such as Wick ordering, symmetric ordering and normal ordering, of which the most significant are the Hermitean orderings, see the discussion by Wolf [3] and also by Bender and Dunne [4]. Weyl-ordered polynomials possess many properties which make them useful for solving a variety of quantum mechanical problems, as we discuss in section 6.

The aim of this paper is to develop properties of Weyl-ordered polynomials in P, Q in fractional-dimensional quantum mechanics, where the defining relations for P, Q are given by the following R-deformed Heisenberg relations:

$$[Q, P] = \mathrm{i}(1 + \nu R), \qquad \{Q, R\} = 0 = \{P, R\}, \qquad R^2 = 1 \tag{1}$$

together with the Hermiticity properties

$$Q^* = Q, \qquad P^* = P, \qquad R^* = R.$$

R is the reflection operator and ν is a Hermitean operator which commutes with each of P, Q, R and so, in effect, may be regarded as a real number. For $\nu = 0$ our results reduce to those well known for the canonical commutation relations. The relevance of relations (1) to quantization in fractional dimensions is discussed in [5], but we summarize the main properties and other details, such as the connection to paraboson quantization, in section 2.

In section 3 we derive general symmetry properties and other identities satisfied by Weyl-ordered polynomials, such as recurrence relations, from a defining generating function in which our results apply to any two operators P, Q. However when P, Q satisfy the specific relations (1) we are able to use Lie-algebraic properties to identify Weyl-ordered polynomials as tensor operators with respect to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, which leads to group transformation properties as well as further recurrence relations (see section 4). Many of these relations are a consequence of the expression of Weyl-ordered polynomials as terminating hypergeometric ${}_3F_2(1)$ functions, multiplied by powers of either P or Q, which we derive in sections 4.3–4.5. A fundamental property is the product formula, derived in section 5, which also has a Lie-algebraic interpretation and which we prove directly by induction. This formula involves a set of polynomials in νR, several properties of which we develop in detail. If we view Weyl-ordered polynomials as a set of fundamental fields in zero-dimensional quantum field theory, then the product formula can be seen as an operator product expansion in the context of paraboson quantization.

The properties we develop reduce to previously known formulae for $\nu = 0$ but are considerably more complicated for general ν, essentially because relations (1) form an infinite-dimensional algebra. For example, the right-hand side of the relation $[Q, P] = \mathrm{i}(1+\nu R)$ does not close under repeated commutation with P or Q which, as a consequence, precludes the general use of simple BCH identities to reorder operator products and also other calculational devices well known within canonical quantization.

We discuss briefly various applications of Weyl-ordered polynomials in section 6, where we use the property that quantum mechanical operators may be expanded in a possibly infinite series of Weyl-ordered polynomials. Because such series may not converge, of main interest here are the applications for which this expansion terminates.

2. Quantization in fractional dimensions

The relations (1) lead to quantization in fractional dimensions as a consequence of the following representations, discussed in [5]. We allow the operators P, Q, R to act in a Hilbert space $\mathfrak{H}$, consisting of differentiable functions $\psi(x)$ defined on $\mathbb{R}$, according to

$$
\begin{aligned}
P\psi(x) &= \left[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{i}\nu}{2}x^{-1}R - \frac{\mathrm{i}(d-1)}{2}x^{-1}\right]\psi(x) \\
&= -\mathrm{i}\psi'(x) + \frac{\mathrm{i}\nu}{2x}\psi(-x) - \frac{\mathrm{i}(d-1)}{2x}\psi(x), \\
Q\psi(x) &= x\psi(x), \\
R\psi(x) &= \psi(-x),
\end{aligned}
\tag{2}
$$

where $d > 0$ is the dimension, and where we may regard x as the radial coordinate r extended to negative values. Relations (1) are satisfied. The inner product in $\mathfrak{H}$ is given by

$$
(\psi, \phi) = \int_{-\infty}^{\infty} |x|^{d-1}\overline{\psi(x)}\phi(x)\,\mathrm{d}x, \tag{3}
$$

and the operators P, Q, R are then Hermitean provided elements ψ of $\mathfrak{H}$ satisfy various conditions, such as suitable behaviour both near the origin and for large x.

If we consider models in which the Hamiltonian H commutes with R, then the wavefunctions ψ_ℓ are either even or odd and also carry a quantum number $\ell = 0, 1, 2, \ldots,$ which we identify as angular momentum (in any dimension d) which is correspondingly even or odd. Hence we have

$$
R\psi_\ell = (-1)^\ell\psi_\ell,
$$

and we identify ν as the operator given by

$$
\nu\psi_\ell = [d - 2 + 2\ell + (-1)^\ell]\psi_\ell.
$$

Since the angular momentum commutes with each of P, Q, R (recalling that Q is represented by the radial coordinate x, which is rotationally invariant) it follows that ν also commutes with each of P, Q, R.

From the representation (2) we obtain

$$
-P^2\psi_\ell = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(d-1)}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\ell(\ell+d-2)}{x^2}\right]\psi_\ell
$$

showing that $-P^2$ reproduces all terms of the radial Laplacian operator, and is defined in any (possibly fractional) dimension $d > 0$. The formulation of Schrödinger wave mechanics in fractional dimensions has been discussed by Stillinger [6] and has been applied to various physical models, usually to Hamiltonians of the form $H = P^2 + V(Q)$, where V is a radial potential, see [5] for further discussion. An algebraic approach to solving such models proceeds via relations (1), although d and the quantized angular momentum ℓ appear only within the representations (2).

Our development here is restricted therefore to the algebra (1). These relations, which have also been studied in other contexts, see, for example, Vasiliev [7] and also [8], originate from the quantization scheme considered by Wigner [9] and subsequently by Yang [10], and are related to the paraboson relations introduced by Green [11]. If we define annihilation and creation operators $a, a^\dagger$ in the usual way according to

$$
a = \tfrac{1}{\sqrt{2}}(Q + \mathrm{i}P), \qquad a^\dagger = \tfrac{1}{\sqrt{2}}(Q - \mathrm{i}P),
$$

then $a, a^\dagger$ satisfy

$$
[a, a^\dagger] = 1 + \nu R, \qquad \{a, R\} = 0 = \{a^\dagger, R\}, \qquad R^2 = 1, \tag{4}
$$

and hence satisfy the trilinear relations of a paraboson algebra with one degree of freedom:

$$
[\{a, a^\dagger\}, a] = -2a, \qquad [\{a, a^\dagger\}, a^\dagger] = 2a^\dagger.
$$

Relations (4) are identical to the defining relations (1) if we identify symbols according to $Q \leftrightarrow a^\dagger$, $P \leftrightarrow -\mathrm{i}a$ and hence our considerations, apart from those which follow from Hermiticity properties, apply directly also to paraboson operators. Weyl-ordered polynomials with respect to the R-deformed algebra (1), or the paraboson algebra (4), have not previously been investigated to our knowledge.

3. Definition and fundamental properties of Weyl-ordered polynomials

The Weyl rule replaces a polynomial in the classical variables p, q by the fully symmetrized, averaged, sum of monomials in P and Q. We denote the Weyl-ordered form of $p^m q^n$ by $T_{mn}(P, Q)$, or simply T_{mn}, following the notation of Bender and Dunne [12]. For any two operators P, Q we have, for example

$$\begin{aligned}
T_{11} &= \tfrac{1}{2}(PQ + QP) \\
T_{21} &= \tfrac{1}{3}(P^2Q + PQP + QP^2) \\
T_{31} &= \tfrac{1}{4}(P^3Q + P^2QP + PQP^2 + QP^3) \\
T_{22} &= \tfrac{1}{6}(P^2Q^2 + PQPQ + PQ^2P + QP^2Q + QPQP + Q^2P^2),
\end{aligned} \tag{5}$$

and, generally, $T_{m0} = P^m$ and $T_{0n} = Q^n$.

A generating function, which can be taken to define $T_{mn}(P, Q)$, is given by

$$(\eta P + \xi Q)^r = \sum_{k=0}^{r} \binom{r}{k} \eta^{r-k} \xi^k T_{r-k,k}(P, Q), \tag{6}$$

where η, ξ are real or complex parameters. This definition corresponds to that of Weyl [2] as we show below. From this expression follows:

$$\exp[t(\eta P + \xi Q)] = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{r} \binom{r}{k} \eta^{r-k} \xi^k T_{r-k,k}(P, Q) \tag{7}$$

for real or complex t.

Since the left-hand side of equation (6) is invariant under the simultaneous interchanges

$$P \longleftrightarrow Q, \qquad \eta \longleftrightarrow \xi,$$

we deduce the symmetry

$$T_{mn}(P, Q) = T_{nm}(Q, P). \tag{8}$$

Weyl-ordered polynomials transform linearly under linear transformations of the operators P, Q. If

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}, \tag{9}$$

where a, b, c, d are in general complex, then

$$T_{mn}(P', Q') = \sum_{k=0}^{m+n} D^k_{mn}(A) T_{m+n-k,k}(P, Q), \tag{10}$$

where the coefficients $D^k_{mn}(A)$ depend on the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ according to

$$D^k_{mn}(A) = n!m! \sum_r \frac{a^r b^{m-r} c^{n+m-k-r} d^{k-m+r}}{r!(m-r)!(n+m-k-r)!(k-m+r)!},$$

and may be expressed in terms of the hypergeometric function, as we show later in equation (53). The result (10) follows from (6) by writing $\eta P' + \xi Q' = \eta' P + \xi' Q$ where

$$(\eta' \quad \xi') = (\eta \quad \xi)\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and expanding η', ξ' in terms of η, ξ on the right-hand side of (6) using the binomial theorem.

The symmetry (8) corresponds to the special case $a = 0 = d$, $b = 1 = c$. Another special case is the scaling property corresponding to $a = \lambda$, $b = 0 = c$, $d = \mu$:

$$T_{mn}(\lambda P, \mu Q) = \lambda^m \mu^n T_{mn}(P, Q).$$

If P, Q are Hermitean then it follows from (6), where now η, ξ are real, that Weyl-ordered polynomials are also Hermitean:

$$T_{mn}(P, Q) = T_{mn}(P, Q)^*. \tag{11}$$

From the identity

$$(\eta P + \xi Q)^{m+n} = (\eta P + \xi Q)^r (\eta P + \xi Q)^{m+n-r}$$

for any integer $0 \leqslant r \leqslant m + n$ there follows, with the help of (6), the convolution formula:

$$\binom{m+n}{m} T_{mn} = \sum_k \binom{r}{k} \binom{m+n-r}{n-k} T_{r-k,k} T_{m+k-r,n-k}, \tag{12}$$

where the sum is over integers k with limits given by

$$\max(0, r - m) \leqslant k \leqslant \min(n, r). \tag{13}$$

This relation reduces to Vandermonde's convolution when P, Q commute. A recursive relation which generates all polynomials T_{mn} for $m, n \geqslant 0$ beginning with $T_{00} = 1$ is

$$\begin{aligned}(m+n)T_{mn} &= mT_{m-1,n}P + nT_{m,n-1}Q \\ &= mPT_{m-1,n} + nQT_{m,n-1},\end{aligned} \tag{14}$$

which is a special case of the identity (12), obtained by choosing $r = m + n - 1$ in which case $k = n - 1, n$ or $r = 1$ in which case $k = 0, 1$.

Weyl ordering may be extended to polynomials of inverse powers of P, Q although relation (14) does not extend directly as it stands to negative values of m or n. For example $n = 1 = -m$ leads to $T_{-2,1}(P, Q) = P^{-1}QP^{-1}$ which is not symmetrized. However, we may define for all $m, n \geqslant 0$ the fully symmetrized functions

$$\begin{aligned}T_{-m,n}(P, Q) &= T_{mn}(P^{-1}, Q) \\ T_{m,-n}(P, Q) &= T_{mn}(P, Q^{-1}) \\ T_{-m,-n}(P, Q) &= T_{mn}(P^{-1}, Q^{-1}).\end{aligned}$$

The recursive formula (14) can be modified to include these cases.

Weyl's original definition [2] of the symmetrized quantum mechanical function $F(P, Q)$ which corresponds to a classical function $f(p, q)$ is stated in terms of the Fourier transform. If $f(p, q)$ is expressed as a Fourier integral,

$$f(p, q) = \int\int \mathrm{e}^{\mathrm{i}(\sigma p + \tau q)} \zeta(\sigma, \tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau,$$

then $F(P, Q)$ is given by

$$F(P, Q) = \int\int \mathrm{e}^{\mathrm{i}(\sigma P + \tau Q)} \zeta(\sigma, \tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau. \tag{15}$$

This definition accords with (6) for the case of multinomials, for which $f(p,q) = p^m q^n$, for then we have

$$\zeta(\sigma,\tau) = \mathrm{i}^{m+n}\delta^{(m)}(\sigma)\delta^{(n)}(\tau) = \overline{\zeta}(-\sigma,-\tau). \tag{16}$$

It follows from (7) that

$$\mathrm{e}^{\mathrm{i}(\sigma P+\tau Q)} = \sum_{r=0}^{\infty} \mathrm{i}^r \sum_{k=0}^{r} \frac{\sigma^{r-k}\tau^k}{k!(r-k)!} T_{r-k,k}(P,Q).$$

By substituting this expression and equation (16) into (15) we obtain $F(P,Q) = T_{mn}(P,Q)$, i.e. the quantum mechanical function corresponding to $f(p,q) = p^m q^n$ is precisely the Weyl-ordered polynomial $T_{mn}(P,Q)$.

The considerations so far, i.e. expressions (5)–(14), apply regardless of the commutation relations satisfied by P, Q.

4. Weyl ordering in fractional-dimensional quantum mechanics

In general Weyl-ordered polynomials can be simplified when specific relations are satisfied by P, Q, in particular if $[Q,P] = \mathrm{i}$ we have

$$\begin{aligned} T_{mn}^{(\nu=0)}(P,Q) &= \frac{1}{2^m}\sum_{r=0}^{m}\binom{m}{r} P^r Q^n P^{m-r} \\ &= \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r} Q^r P^m Q^{n-r}, \end{aligned} \tag{17}$$

which was derived by McCoy [13] in 1932. This expression follows from the relation

$$\mathrm{e}^{\eta P+\xi Q} = \mathrm{e}^{\frac{1}{2}\eta P}\,\mathrm{e}^{\xi Q}\,\mathrm{e}^{\frac{1}{2}\eta P}$$

(for $\nu = 0$) by comparing coefficients of $\eta^m\xi^n$ in the expansion of the exponentials on each side, and with the help of the generating function (6).

Formula (17) was rederived by Biedenharn and Louck [14] (p 254, see also [15]) and again by Bender and Dunne [12]. The results of [14, 15] are presented in the language of boson creation and annihilation operators but because most of the results do not depend on any Hermiticity properties are applicable to any Heisenberg algebra, and hence to quantum mechanical operators P, Q by identifying $Q \leftrightarrow a^\dagger$, $P \leftrightarrow -\mathrm{i}a$. The same remark applies to our development, by identifying P, Q with the corresponding symbols for paraboson operators.

Further properties of Weyl-ordered polynomials for $\nu = 0$ are summarized in [16, 17] (see section 8.3, p 261), including the expression of T_{mn} in terms of hypergeometric functions, specifically:

$$T_{mn}^{(\nu=0)}(P,Q) = 2^{-m}(-\mathrm{i})^n(N-n+1)_n\,{}_2F_1\left(\begin{matrix}N+1,-m\\ N-n+1\end{matrix};-1\right)P^{m-n} \tag{18}$$

where we have used the Pochhammer notation:

$$(\alpha)_r = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+r-1) & \text{for} \quad r>0 \\ 1 & \text{for} \quad r=0 \\ \dfrac{1}{(\alpha-1)(\alpha-2)\cdots(\alpha+r)} & \text{for} \quad r<0, \end{cases} \tag{19}$$

for any integers r. The operator N in the above equation is defined, for general ν, by

$$N = \frac{\mathrm{i}}{2}(PQ+QP) - \frac{1}{2} = \mathrm{i}T_{11} - \frac{1}{2}, \tag{20}$$

and satisfies

$$[N, Q] = Q, \qquad [N, P] = -P. \tag{21}$$

As a result of the property (18) Weyl-ordered polynomials for $\nu = 0$ inherit many of the properties of hypergeometric functions, for example the symmetry (8) follows from Euler's linear transformation formula, and relations such as (14) are equivalent to contiguity relations of hypergeometric functions. Some of these properties were observed also in [18, 19].

Let us now consider the polynomials $T_{mn}(P, Q)$ for general ν, where P, Q now satisfy the algebra (1). Although these relations involve a third operator R we need consider Weyl-ordering with respect to P, Q only, since R anticommutes with each of P, Q and so can be moved to the right of any functions of P, Q. In particular we have $RT_{mn} = (-1)^{m+n}T_{mn}R$.

From equations (1) we find

$$[Q^2, P] = 2\mathrm{i}Q, \qquad [P^2, Q] = -2\mathrm{i}P, \tag{22}$$

which in fact we may take as defining relations, and were proposed by Wigner [9] in connection with quantization of the harmonic oscillator. As a consequence of (22) many properties of Weyl-ordered polynomials do not depend specifically on ν, as we show in following sections.

Also as a consequence of (22), the operator N defined by (20) satisfies (21) for general ν. It is convenient to note here the invariance of relations (1) and (22) under the interchange

$$P \longleftrightarrow Q, \qquad \mathrm{i} \longleftrightarrow -\mathrm{i}, \tag{23}$$

in which case $N \longrightarrow -N - 1$; this symmetry is a special case of the symmetries discussed in section 4.6.

4.1. Tensor operator properties

The operators

$$J_0 = \frac{\mathrm{i}}{4}(PQ + QP), \qquad J_+ = -\frac{1}{2}Q^2, \qquad J_- = -\frac{1}{2}P^2, \tag{24}$$

generate the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$:

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0.$$

It is more convenient to choose the generators in this nonunitary form rather than the Hermitean linear combinations $\{L_0, L_1, L_2\}$ which generate unitary representations of $\mathfrak{sl}_2(\mathbb{R})$, where

$$L_0 = -\tfrac{1}{4}(QP + PQ), \qquad L_1 = \tfrac{1}{4}(Q^2 + P^2), \qquad L_2 = \tfrac{1}{4}(Q^2 - P^2).$$

It is worth pointing out, although we will not use this property directly, that the representation (24) of $\mathfrak{sl}_2$ can be extended to a non-Hermitean representation of the superalgebra osp(1|2) by defining the fermionic generators

$$V_+ = \frac{\mathrm{i}}{2\sqrt{2}}Q, \qquad V_- = -\frac{1}{2\sqrt{2}}P,$$

which satisfy

$$[J_0, V_\pm] = \pm\tfrac{1}{2}V_\pm, \qquad [J_\pm, V_\mp] = V_\pm,$$

and

$$\{V_+, V_-\} = -\tfrac{1}{2}J_0, \qquad \{V_\pm, V_\pm\} = \pm\tfrac{1}{2}J_\pm.$$

For discussion of representations of osp(1|2), see [20].

As in the case $\nu = 0$ [16] Weyl-ordered polynomials can be viewed as tensor operators with respect to the $\mathfrak{sl}_2(\mathbb{C})$ algebra. If we define

$$\mathcal{P}_{jm}(P, Q) = \frac{\mathrm{i}^{j-m} 2^j}{\sqrt{(j+m)!(j-m)!}} T_{j-m,j+m}(P, Q) \tag{25}$$

for half-integers j, m with $-j \leqslant m \leqslant j$, then we have

$$[J_\pm, \mathcal{P}_{jm}] = \sqrt{(j \mp m)(j \pm m + 1)} \mathcal{P}_{j,m\pm1}, \qquad [J_0, \mathcal{P}_{jm}] = m\mathcal{P}_{jm}. \tag{26}$$

This follows by first writing the generating function (6) in the form

$$(\eta P + \xi Q)^{2j} = \frac{(2j)!}{2^j} \sum_{m=-j}^{j} \mathrm{i}^{-j+m} \Phi_{jm}(\eta, \xi) \mathcal{P}_{jm}(P, Q), \tag{27}$$

where

$$\Phi_{jm}(\eta, \xi) = \frac{\eta^{j-m} \xi^{j+m}}{\sqrt{(j-m)!(j+m)!}},$$

following the conventions in [14, 16]. The normalized monomials Φ_{jm} span a linear vector space which carries a finite-dimensional representation of the $\mathfrak{sl}_2(\mathbb{C})$ algebra generated by $K_\pm$, K_0, defined by

$$K_+ = \xi \frac{\partial}{\partial \eta}, \qquad K_- = \eta \frac{\partial}{\partial \xi}, \qquad K_0 = \frac{1}{2}\left(\xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta}\right),$$

with matrix elements given by

$$K_0 \Phi_{jm} = m \Phi_{jm}, \qquad K_\pm \Phi_{jm} = \sqrt{(j \mp m)(j \pm m + 1)} \Phi_{j,m\pm1}. \tag{28}$$

With the help of the formula

$$[A, B^n] = \sum_{r=0}^{n-1} B^r [A, B] B^{n-r-1},$$

which is valid for positive integers n and any operators A, B, we find

$$[J_\pm, (\eta P + \xi Q)^{2j}] = \mp \mathrm{i} K_\mp (\eta P + \xi Q)^{2j},$$
$$[J_0, (\eta P + \xi Q)^{2j}] = K_0 (\eta P + \xi Q)^{2j}.$$

Equations (26) now follow after using the generating function (27).

For any given j all components of the tensor operator $\mathcal{P}_{jm}$ can be calculated from the component $\mathcal{P}_{jj} = \frac{2^j}{\sqrt{(2j)!}} Q^{2j}$ of highest weight by repeated commutation with the lowering generator J_-, using (26):

$$\mathcal{P}_{jm} = \left[\frac{(j+m)!}{(2j)!(j-m)!}\right]^{\frac{1}{2}} [J_-, \mathcal{P}_{jj}]_{(j-m)},$$

where we define the repeated commutator recursively by

$$[A, B]_{(n)} = [A, [A, B]_{(n-1)}], \qquad [A, B]_{(0)} = B. \tag{29}$$

Similarly all components can be generated by repeated commutation with J_+ beginning with the component $\mathcal{P}_{j,-j}$ of lowest weight. In particular, the pair

$$\left(\mathcal{P}_{\frac{1}{2}\frac{1}{2}}, \mathcal{P}_{\frac{1}{2}-\frac{1}{2}}\right) = \sqrt{2}(Q, \mathrm{i}P)$$

forms a spinor operator with respect to the $\mathfrak{sl}_2(\mathbb{C})$ algebra.

Further properties of $\mathcal{P}_{jm}$ also follow, for example any two Weyl-ordered polynomials can be vector coupled using Clebsch–Gordan coefficients to produce another Weyl-ordered polynomial, which constitutes a product law which we consider in more detail in section 5.

In terms of T_{mn} relations (26) may be stated in the form:

$$[P^2, T_{mn}] = -2\mathrm{i}nT_{m+1,n-1}, \qquad [Q^2, T_{mn}] = 2\mathrm{i}mT_{m-1,n+1}. \tag{30}$$

It is also useful to note the following commutators:

$$\begin{aligned}
[P, T_{mn}] &= -\mathrm{i}\frac{n}{n+m}T_{m,n-1}\left[n+m+\frac{1}{2}(1-(-1)^{n+m})\nu R\right] \\
[Q, T_{mn}] &= \mathrm{i}\frac{m}{n+m}T_{m-1,n}\left[n+m+\frac{1}{2}(1-(-1)^{n+m})\nu R\right],
\end{aligned} \tag{31}$$

which can be derived directly from the generating function (6) using

$$[Q, (\eta P + \xi Q)^r] = \mathrm{i}\eta(\eta P + \xi Q)^{r-1}\left[r + \tfrac{1}{2}(1-(-1)^r)\nu R\right], \tag{32}$$

which is proved by induction on r. The second of the relations in each of (30) and (31) follow from the first with the help of the symmetries (8) and (23).

4.2. *Specific form for Weyl-ordered polynomials*

The McCoy form (17) of Weyl-ordered polynomials applies only to the canonical commutation relations, but for general ν we find, for example,

$$T_{12}(P, Q) = \frac{1}{2}(Q^2P + PQ^2) - \frac{\mathrm{i}}{3}\nu QR,$$

where the last term is additional to the McCoy form. A useful form which is valid for all ν can, however, be derived as follows.

From the commutator $[Q^2, P] = 2\mathrm{i}Q$ and with the help of the BCH formula

$$\mathrm{e}^A B\,\mathrm{e}^{-A} = \sum_n \frac{1}{n!}[A, B]_{(n)},$$

where the repeated commutator is defined in (29), we obtain

$$\mathrm{e}^{-\mathrm{i}tQ^2} P\,\mathrm{e}^{\mathrm{i}tQ^2} = P + 2tQ,$$

which describes the linear transformation by a group element of the spinor component P. From this follows $\mathrm{e}^{-\mathrm{i}tQ^2}\mathrm{e}^{sP}\mathrm{e}^{\mathrm{i}tQ^2} = \mathrm{e}^{sP+2stQ}$ for any parameters s, t and hence

$$\begin{aligned}
\mathrm{e}^{\eta P+\xi Q} &= \mathrm{e}^{-\frac{\mathrm{i}\xi}{2\eta}Q^2}\,\mathrm{e}^{\eta P}\,\mathrm{e}^{\frac{\mathrm{i}\xi}{2\eta}Q^2} \\
&= \mathrm{e}^{\frac{\mathrm{i}\eta}{2\xi}P^2}\,\mathrm{e}^{\xi Q}\,\mathrm{e}^{-\frac{\mathrm{i}\eta}{2\xi}P^2}.
\end{aligned} \tag{33}$$

We now use the generating function (7) to expand the exponential on the left-hand side and equate coefficients of $\eta^{r-k}\xi^k$ on both sides to obtain

$$\begin{aligned}
T_{mn}(P, Q) &= \frac{\mathrm{i}^n n!m!}{2^n(n+m)!}\sum_{r=0}^{n}\frac{(-1)^{n-r}}{r!(n-r)!}Q^{2n-2r}P^{m+n}Q^{2r} \\
&= \frac{\mathrm{i}^m n!m!}{2^m(n+m)!}\sum_{r=0}^{m}\frac{(-1)^{r}}{r!(m-r)!}P^{2m-2r}Q^{m+n}P^{2r},
\end{aligned} \tag{34}$$

where we again use the symmetries (23) and (8).

The two expressions (34) for T_{mn}, which are valid for any ν, are polynomials in (P, Q) of degree $(m+n, 2m)$ and $(2m, m+n)$ respectively, whereas T_{mn} is defined as a polynomial in P, Q of degree m, n. For example, we have

$$T_{11} = \frac{\mathrm{i}}{4}[P^2, Q^2], \qquad T_{12} = -\frac{1}{24}[Q^2, [Q^2, P^3]] = \frac{\mathrm{i}}{6}[P^2, Q^3], \tag{35}$$

where we have used both forms shown in equation (34). These expressions follow from the tensor operator properties of T_{mn}, in particular the fact that all components T_{mn} can be generated by repeated commutation of P^2 with the highest weight component $T_{0,m+n} = Q^{m+n}$, or similarly by repeated commutation with Q^2 from the lowest weight component $T_{m+n,0} = P^{m+n}$. In general we have

$$\begin{aligned} T_{mn}(P, Q) &= \frac{m!(-\mathrm{i})^n}{2^n(n+m)!}[Q^2, P^{m+n}]_{(n)} \\ &= \frac{n!\mathrm{i}^m}{2^m(n+m)!}[P^2, Q^{m+n}]_{(m)}. \end{aligned}$$

4.3. Special function form

Now we generalize the special function form (18) of Weyl-ordered polynomials, which provides a convenient form for their manipulation, by first writing them in terms of a function F_{mn} to be defined, which we then show can be identified with a ${}_3F_2(1)$ hypergeometric function. First, we establish the following identities, for any $r \in \mathbb{N}$:

$$\begin{aligned} Q^r P^r &= (-\mathrm{i})^r \prod_{k=1}^{r} \left[N + k - r - \frac{1}{2}(-1)^{r-k}\nu R\right] \\ P^r Q^r &= \mathrm{i}^r \prod_{k=1}^{r} \left[-N + k - r - 1 - \frac{1}{2}(-1)^{r-k}\nu R\right], \end{aligned} \tag{36}$$

where N is defined by (20) and satisfies (21) for general ν. These equations are proved by induction on r, noting that the second follows from the first by the symmetry (23), under which $N \to -N-1$. For notational convenience let us generalize (19) by defining

$$(\alpha, \lambda)_r = \begin{cases} \displaystyle\prod_{k=1}^{r} \left[\alpha + k - 1 - (-1)^k \frac{\lambda}{2}\right] & \text{for} \quad r > 0 \\ 1 & \text{for} \quad r = 0 \\ \displaystyle\left(\prod_{k=1}^{-r} \left[\alpha - k + (-1)^k \frac{\lambda}{2}\right]\right)^{-1} & \text{for} \quad r < 0, \end{cases} \tag{37}$$

for any integers r and any real numbers α, λ. In particular $(\alpha, 0)_r = (\alpha)_r$. Then we have

$$\begin{aligned} Q^r P^r &= (-\mathrm{i})^r (N - r + 1, (-1)^r \nu R)_r \\ P^r Q^r &= \mathrm{i}^r (-N - r, (-1)^r \nu R)_r, \end{aligned} \tag{38}$$

which are each valid for both positive and negative integers r.

The form (34) for T_{mn} can now be expressed as a function of the commuting operators N and νR multiplied by powers of either P or Q, by grouping factors suitably in (34), for example

$$Q^{2n-2r} P^{m+n} Q^{2r} = (Q^{2n-2r} P^{2n-2r})(P^{m-n+2r} Q^{m-n+2r}) Q^{n-m},$$

and substituting from (38). Let us define a real-valued rational function $F_{mn}(x, y)$ of real variables x, y for any non-negative integers m, n according to

$$F_{mn}(x, y) = \frac{n!m!}{2^n(n+m)!} \sum_{r=0}^{n} \frac{(-1)^r}{r!(n-r)!}(x - 2n + 2r + 1, y)_{2n-2r} \\ \times (-x - m + n - 2r, y(-1)^{m-n})_{m-n+2r}, \tag{39}$$

then equations (34) may be written

$$T_{mn}(P, Q) = \mathrm{i}^m F_{mn}(N, \nu R) Q^{n-m} \\ = (-\mathrm{i})^n F_{nm}(-N-1, \nu R) P^{m-n}. \tag{40}$$

A specific example which we state for future reference is

$$T_{1n}(P, Q) = \mathrm{i} F_{1n}(N, \nu R) Q^{n-1} \\ = \mathrm{i}\left[-1 + \frac{n}{2} - N - \frac{1}{4(n+1)}(1 + (-1)^n)\nu R\right] Q^{n-1}, \tag{41}$$

where we have used

$$F_{1n}(x, y) = -1 + \frac{n}{2} - x - \frac{y}{4(n+1)}(1 + (-1)^n). \tag{42}$$

This expression may be calculated from the recurrence relation (47) below.

4.4. Special function properties

Let us summarize various properties of $F_{mn}(x, y)$, some of which are not obvious from the explicit definition (39). Since $T_{m0} = P^m$ we have

$$F_{0n}(x, y) = 1$$

for all x, y and, directly from the definition,

$$F_{m0}(x, y) = (-x - m, y(-1)^m)_m.$$

By equating the right-hand sides of (40) we deduce (setting $x = N$ and $y = \nu R$):

$$F_{mn}(x, y) = (-1)^n(-x - m + n, (-1)^{m-n} y)_{m-n} F_{nm}(-x - 1, y), \tag{43}$$

which is equivalent to the symmetry (11).

By comparison with equation (18) we find

$$F_{mn}(x, 0) = 2^{-n}(-x - m)_m \, {}_2F_1\left(\begin{matrix} -x, -n \\ -x - m \end{matrix}; -1\right). \tag{44}$$

On the other hand, we find directly from definition (39) that $F_{mn}(x, 0)$ may be expressed as a generalized ${}_3F_2$ hypergeometric function (after some manipulation):

$$F_{mn}(x, 0) = \frac{m!(-1)^{n-m}(x + 1 - 2n)_{n+m}}{2^n(n+m)!} {}_3F_2\left(\begin{matrix} -n, \frac{1}{2}(1 + x + m - n), \frac{1}{2}(2 + x + m - n) \\ \frac{1}{2}(1 + x - 2n), \frac{1}{2}(2 + x - 2n) \end{matrix}; 1\right). \tag{45}$$

This ${}_3F_2$ function can be transformed into the ${}_2F_1$ function shown in (44) with the help of a transformation due to Whipple (see [21], equation (3.7)).

The commutators (30) imply the following recurrence relations:

$$2m F_{m-1,n+1}(x, y) = F_{mn}(x - 2, y) - F_{mn}(x, y), \tag{46}$$

and

$$2nF_{m+1,n-1}(x,y) = -\Big(x+\frac{y}{2}+1\Big)\Big(x-\frac{y}{2}+2\Big)F_{mn}(x+2,y) + \left[x+\frac{(-1)^{n-m}y}{2}-n+m+1\right]\left[x-\frac{(-1)^{n-m}y}{2}-n+m+2\right]F_{mn}(x,y), \tag{47}$$

of which the last relation is significant in that it defines $F_{mn}(x,y)$ uniquely for all m, n, x, y beginning with $F_{0n}(x,y) = 1$. In particular this relation provides a convenient method for computing $F_{mn}(x,y)$ explicitly for generic n, x, y and specific m by means of a computer algebra system. Such calculations show that $F_{mn}(x,y)$ is a polynomial: (1) in x of exactly degree m, where the coefficient of x^m is $(-1)^m$; (2) in y of degree at most m; (3) in x, y of combined degree exactly m.

Other recurrence relations follow from the commutators (32) and also the recursive formulae (14). These latter formulae all follow from the convolution formula (12) which in terms of $F_{mn}(x,y)$ reads

$$\binom{m+n}{m}F_{mn}(x,y) = \sum_k \binom{r}{k}\binom{m+n-r}{n-k}F_{r-k,k}(x,y)F_{m+k-r,n-k}(x+r-2k,(-1)^r y), \tag{48}$$

where the summation index k satisfies (13).

In applications of Weyl-ordered polynomials it is sometimes necessary to extend the definition of T_{mn} to negative values of m, n, as explained by Bender and Dunne [12]. The two expressions (34) for T_{mn} are each valid for $m+n < 0$, in the first case for $m < 0$ and $n \geqslant 0$ by writing $(n+m)!/m! = (m+1)_n$, and for $n < 0$, $m \geqslant 0$ in the second case. Similarly $F_{mn}(x,y)$ as defined by (39) can be extended to negative values of either m or n provided $m+n < 0$. The recurrence relation (46) and the symmetry (43) can then be used to generate $F_{mn}(x,y)$ for negative m or n without the restriction $m+n < 0$, where we use $F_{1,-1}(x,y) = -\frac{3}{2} - x$, as follows from (42), and

$$F_{-1,1}(x,y) = \frac{-x+\frac{1}{2}}{\big(x-1+\frac{y}{2}\big)\big(x-\frac{y}{2}\big)}.$$

4.5. Expression as hypergeometric polynomials

Expression (45) for $F_{mn}(x,0)$ may be generalized in that $F_{mn}(x,y)$ can also be expressed in terms of a hypergeometric function. This is achieved by writing the product $(\alpha,\lambda)_r$ defined in (37) in terms of the standard Pochhammer symbol, defined in (19), as follows:

$$\begin{aligned}(\alpha,\lambda)_{2r} &= 2^{2r}\left(\frac{\alpha}{2}+\frac{\lambda}{4}\right)_r\left(\frac{\alpha+1}{2}-\frac{\lambda}{4}\right)_r\\ (\alpha,\lambda)_{2r+1} &= 2^{2r+1}\left(\frac{\alpha}{2}+\frac{\lambda}{4}\right)_{r+1}\left(\frac{\alpha+1}{2}-\frac{\lambda}{4}\right)_r,\end{aligned}$$

where r is an integer.

By writing all factors in definition (39) of $F_{mn}(x,y)$ in terms of Pochhammer products we are able to identify $F_{mn}(x,y)$ as a terminating ${}_3F_2(1)$ function, with separate expressions for the cases in which $m-n$ is even or odd. We may combine these expressions using the notation of the ceiling function $\lceil l \rceil$, and the floor function $\lfloor l \rfloor$ for integers and half odd integers l, each defined by

$$\lceil l \rceil = \begin{cases} l & \text{for} \quad \text{integers } l \\ l + \frac{1}{2} & \text{for} \quad \text{half odd integers } l, \end{cases}$$
$$\lfloor l \rfloor = \begin{cases} l & \text{for} \quad \text{integers } l \\ l - \frac{1}{2} & \text{for} \quad \text{half odd integers } l. \end{cases} \tag{49}$$

We have $\lceil l \rceil + \lfloor l \rfloor = 2l$ for all integers and half odd integers l.

Then we find

$$F_{mn}(x, y) = \frac{(-1)^{m-n} 2^m}{(m+1)_n} \left(-\frac{x}{2} - \frac{y}{4} + \frac{1}{2}\right)_n \left(-\frac{x}{2} + \frac{y}{4}\right)_n$$
$$\times \left(\frac{x}{2} - \frac{y}{4} + 1\right)_{\lfloor \frac{m-n}{2} \rfloor} \left(\frac{x}{2} + \frac{y}{4} + \frac{1}{2}\right)_{\lfloor \frac{m-n+1}{2} \rfloor}$$
$$\times {}_3F_2 \left(\begin{matrix} -n, \frac{x}{2} + \frac{y}{4} + \frac{1}{2} + \lfloor \frac{m-n+1}{2} \rfloor, \frac{x}{2} - \frac{y}{4} + \lceil \frac{m-n+1}{2} \rceil \\ \frac{x}{2} + \frac{y}{4} + \frac{1}{2} - n, \frac{x}{2} - \frac{y}{4} + 1 - n \end{matrix} ; 1 \right). \tag{50}$$

This terminating ${}_3F_2$ hypergeometric function is $(-m)$-balanced (as defined in [22]) since the sum of the numerator parameters minus the sum of the denominator parameters is m. If $m - n$ is even then this function defines a nearly poised series of the first kind [22]. The operator identities satisfied by Weyl-ordered polynomials are therefore related to properties of this ${}_3F_2$ function, for example the recurrence relations (46), (47) are consequences of three-term relations satisfied by ${}_3F_2(1)$ functions. These relations have been known since the work of Thomae (1879) and Whipple (1925) and are discussed by Slater [23] (section 4.3). In particular (46) follows from the identity:

$${}_3F_2 \left(\begin{matrix} -n, a, b \\ a-c, b-c \end{matrix} ; 1 \right) - {}_3F_2 \left(\begin{matrix} -n-1, a, b \\ a-c, b-c \end{matrix} ; 1 \right)$$
$$= \frac{ab}{(a-c)(b-c)} {}_3F_2 \left(\begin{matrix} -n, a+1, b+1 \\ a-c+1, b-c+1 \end{matrix} ; 1 \right)$$

where n is a non-negative integer, as may be proved directly from the series definition and, similarly, (47) follows from

$$(a-n)(b-n) {}_3F_2 \left(\begin{matrix} -n, a+c, b+c \\ a-n, b-n \end{matrix} ; 1 \right) - ab\, {}_3F_2 \left(\begin{matrix} -n, a+c+1, b+c+1 \\ a-n+1, b-n+1 \end{matrix} ; 1 \right)$$
$$= n(2c+n+1) {}_3F_2 \left(\begin{matrix} -n+1, a+c+1, b+c+1 \\ a-n+1, b-n+1 \end{matrix} ; 1 \right).$$

4.6. Symmetries

Relations (1) are invariant under the linear transformations (9) where the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is unimodular: $\det A = 1 = ad - bc$. Since P, Q are Hermitean we also require that A be a real matrix, i.e. A is an element of the Lie group $\mathrm{SL}_2(\mathbb{R})$. Weyl-ordered polynomials transform linearly under such transformations, as shown in equation (10).

The tensor operator components $\mathcal{P}_{jm}$ defined by (25) transform as finite-dimensional representations of $\mathrm{SL}_2(\mathbb{R})$ as we now show. From (27) we have

$$\sum_{m=-j}^{j} \mathrm{i}^{-j+m} \Phi_{jm}(\eta, \xi) \mathcal{P}_{jm}(P', Q') = \sum_{m=-j}^{j} \mathrm{i}^{-j+m} \Phi_{jm}(\eta', \xi') \mathcal{P}_{jm}(P, Q) \tag{51}$$

where

$$\eta P' + \xi Q' = \eta' P + \xi' Q = (\eta \quad \xi) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}.$$

We find directly that the basis vectors Φ_{jm} transform according to

$$\Phi_{jm}(\eta', \xi') = \sum_{m'=-j}^{j} D^j_{m'm}(A) \Phi_{jm'}(\eta, \xi) \tag{52}$$

where

$$D^j_{m'm}(A) = \sqrt{(j-m)!(j+m)!(j-m')!(j+m')!} \times \sum_r \frac{a^r b^{j-m'-r} c^{j-m-r} d^{m+m'+r}}{r!(j-m'-r)!(j-m-r)!(m+m'+r)!}$$

and where the summation index r satisfies $\max(0, -m-m') \leqslant r \leqslant \min(j-m, j-m')$. These matrix elements also have an expression as hypergeometric functions, namely:

$$D^j_{m'm}(A) = \frac{b^{j-m'} c^{j-m} d^{m+m'}}{(m+m')!} \sqrt{\frac{(j+m)!(j+m')!}{(j-m)!(j-m')!}} \, {}_2F_1 \left(\begin{matrix} -j+m, -j+m' \\ m+m'+1 \end{matrix} ; \frac{ad}{bc} \right). \tag{53}$$

The matrices $D^j(A)$ are nonunitary representation matrices of dimension $2j+1$ of $\mathrm{SL}_2(\mathbb{R})$, in fact they satisfy $D^j(A)D^j(B) = D^j(AB)$ for any complex matrices A, B, as follows from (52), see also [24] p 217. We deduce from (51) that the tensor operator components transform according to

$$\mathcal{P}_{jm}(P', Q') = \sum_{m'=-j}^{j} \mathrm{i}^{m'-m} D^j_{mm'}(A) \mathcal{P}_{jm'}(P, Q).$$

Special cases of this transformation are $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, corresponding to the interchange of P, Q, and $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, corresponding to the rescaling of P, Q. This leads to

$$\mathcal{P}_{jm}(Q, -P) = \mathrm{i}^{-2m}(-1)^{j+m} \mathcal{P}_{j,-m}(P, Q)$$
$$\mathcal{P}_{jm}(\lambda P, \lambda^{-1} Q) = \lambda^{-2m} \mathcal{P}_{jm}(P, Q).$$

Besides the continuous symmetries, relations (1) are also invariant under the discrete symmetry

$$R \longrightarrow -R, \qquad \nu \longrightarrow -\nu.$$

5. Product formula

The product of two Weyl-ordered polynomials is also a polynomial in P, Q and hence is expressible as a linear combination of Weyl-ordered polynomials, where in general the coefficients depend linearly on R. This formula, the product law, is a fundamental property which we may use to manipulate Weyl-ordered polynomials, for example we may calculate commutators and anticommutators of any two Weyl-ordered polynomials and hence perform unitary transformations involving Weyl-ordered polynomials. The form of the product law and its properties follow from the interpretation of Weyl-ordered polynomials as tensor operators, as described in section 4.1, and so involves tensor operator couplings through Clebsch–Gordan coefficients. The formula for $\nu = 0$ was given by Biedenharn and Louck [14] (p 247) and also by Bender and Dunne [12], and may be proved by means of the operator identity

$$\mathrm{e}^{\eta P+\xi Q}\,\mathrm{e}^{\eta' P+\xi' Q} = \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi\eta'-\eta\xi')}\,\mathrm{e}^{(\xi+\xi')Q+(\eta+\eta')P}.$$

After expanding each side of this identity using the binomial theorem and substituting for the generating function for Weyl-ordered polynomials, the product formula follows by equating coefficients of powers of η, η', ξ, ξ' on both sides. Since the above operator identity is valid only for $\nu = 0$, our proof for general ν follows instead the induction proof outlined in [14].

The product law for general ν is given by

$$\mathcal{P}_{jm}(P,Q)\mathcal{P}_{j'm'}(P,Q) = \sum_{k=|j'-j|}^{j'+j} \Delta(j,j',k,\nu R) C^{j'jk}_{m'mm+m'} \mathcal{P}_{k,m+m'}(P,Q), \tag{54}$$

where the Clebsch–Gordan coefficients have the standard definition, and where the coefficients $\Delta(j, j', k, \nu R)$ are defined below. From orthogonality of the Clebsch–Gordan coefficients we deduce the inverted product law

$$\Delta(j,j',k,\nu R)\mathcal{P}_{k,m''}(P,Q) = \sum_{m+m'=m''} C^{j'jk}_{m'mm''} \mathcal{P}_{jm}(P,Q)\mathcal{P}_{j'm'}(P,Q), \tag{55}$$

which expresses the standard vector coupling of two tensor operators $\mathcal{P}_{jm}$ and $\mathcal{P}_{j'm'}$ to form a third tensor operator $\mathcal{P}_{k,m''}$, multiplied by a coefficient Δ which is an invariant of the algebra, i.e. is independent of m''. This relation generalizes the convolution formula (12) which applies, however, to any two operators P, Q.

5.1. *Definition and properties of* $\Delta(j,k,l,y)$ *and* $\mathfrak{P}(j,k,l,y)$

The coefficients $\Delta(j, k, l, y)$ are defined only when j, k, l are non-negative integers or half odd integers, and satisfy the triangle conditions, which may be stated in the form

$$j \in \{k+l, k+l-1, \ldots, |k-l|\},$$

and are valid also for any permutations of j, k, l. Two special cases which follow from (54), by putting $j = 0$ or $j' = 0$, are

$$\Delta(0,j,j,y) = 1 = \Delta(j,0,j,y).$$

By taking the Hermitean conjugate of the product law and using

$$\mathcal{P}^*_{jm} = (-1)^{j-m}\mathcal{P}_{jm},$$

together with $R\mathcal{P}_{jm} = \mathcal{P}_{jm}(-1)^{2j}R$ and a symmetry property of the Clebsch–Gordan coefficients, we find

$$\Delta(j,k,l,y) = \Delta(k,j,l,(-1)^{2l}y).$$

For $\nu = 0$ we have (see [14], p 247)

$$\Delta(j,k,l,0) = \frac{\nabla(jkl)}{\sqrt{2l+1}} = \left[\frac{(j+k+l+1)!}{(2l+1)(j+k-l)!(j-k+l)!(-j+k+l)!}\right]^{\frac{1}{2}} \tag{56}$$

where $\nabla(jkl)$ is defined as in [14].

For convenience we also state the following coefficient:

$$\begin{aligned}\Delta\left(\frac{1}{2}, l+\frac{1}{2}, l, y\right) &= \sqrt{2l+2}\left[1 + \frac{1}{2(2l+2)}(1-(-1)^{2l})y + \frac{1}{2(2l+1)}(1+(-1)^{2l})y\right] \\ &= \sqrt{2l+2}\left(1 + \frac{y}{2\lceil l\rceil + 1}\right),\end{aligned} \tag{57}$$

where $\lceil l\rceil$ denotes the ceiling function defined in equation (49).

The general coefficients are defined recursively by

$$\Delta(j,k,l,y) = \left[\frac{(j+k-l)(-j+k+l+1)}{(2j)^2(2l+1)}\right]^{\frac{1}{2}} \Delta\left(j-\frac{1}{2},k,l+\frac{1}{2},-y\right)\Delta\left(\frac{1}{2},l+\frac{1}{2},l,y\right)$$
$$+\left[\frac{(2l)(j-k+l)(j+k+l+1)}{(2j)^2(2l+1)}\right]^{\frac{1}{2}} \Delta\left(j-\frac{1}{2},k,l-\frac{1}{2},-y\right), \quad (58)$$

beginning with $\Delta(0,l,l,y)=1$. For example if $j=\frac{1}{2}$ we must have $l=k\pm\frac{1}{2}$ which leads to

$$\Delta\left(\tfrac{1}{2},k,k+\tfrac{1}{2},y\right)=\sqrt{2k+1}.$$

In general $\Delta(j,k,l,y)$ is a polynomial in y of degree at most $j+k-l$, as follows by induction from (58). Hence $\Delta(j,k,j+k,y)$ is independent of y and so from (56):

$$\Delta(j,k,j+k,y)=\left[\frac{(2j+2k)!}{(2j)!(2k)!}\right]^{\frac{1}{2}}.$$

Let us define the polynomial $\mathfrak{P}(j,k,l,y)$ according to

$$\mathfrak{P}(j,k,l,y)=\frac{\Delta(j,k,l,y)}{\Delta(j,k,l,0)},$$

where $\Delta(j,k,l,0)$ is given in (56), then from (58) this polynomial satisfies the relation

$$\mathfrak{P}(j,k,l,y)=\frac{(j+k-l)}{2j}\mathfrak{P}\left(j-\frac{1}{2},k,l+\frac{1}{2},-y\right)\left(1+\frac{y}{2\lceil l\rceil+1}\right)$$
$$+\frac{(j-k+l)}{2j}\mathfrak{P}\left(j-\frac{1}{2},k,l-\frac{1}{2},-y\right). \quad (59)$$

Hence $\mathfrak{P}(j,k,l,y)$ is a polynomial in y of degree at most $j+k-l$, with rational coefficients, and with $\mathfrak{P}(j,k,l,0)=1$. The degree in y is not necessarily equal to $j+k-l$, for example we find $\mathfrak{P}(1,1,1,y)=1$. We also have the boundary values

$$\mathfrak{P}(j,k,j+k,y)=1,$$

and

$$\mathfrak{P}(j,j+l,l,y)=\frac{\prod_{i=1}^{\lceil j\rceil}(y+2\lceil l\rceil-1+2\mathrm{i})\prod_{i=1}^{\lfloor j\rfloor}(y-2\lfloor l\rfloor-1-2\mathrm{i})}{\prod_{i=1}^{\lceil j\rceil}(2\lceil l\rceil-1+2\mathrm{i})\prod_{i=1}^{\lfloor j\rfloor}(-2\lfloor l\rfloor-1-2\mathrm{i})}, \quad (60)$$

which is of degree $2j$ in y. From the symmetry relation

$$\mathfrak{P}(j,k,l,y)=\mathfrak{P}(k,j,l,(-1)^{2l}y)$$

we also obtain an explicit expression for $\mathfrak{P}(k+l,k,l,y)$.

5.2. *Proof of the product law*

We firstly prove the case $j=1/2$ of the product law (54) for all possible values of j', m, m'. There are four such equations which in terms of T_{mn} are given by, firstly:

$$PT_{mn}=T_{m+1,n}-\frac{\mathrm{i}n}{2}\left[1+\frac{(1-(-1)^{m+n})\nu R}{2(m+n)}+\frac{(1+(-1)^{m+n})\nu R}{2(m+n+1)}\right]T_{m,n-1}, \quad (61)$$

from which the remaining equations follow by means of the commutators (31) or by taking the Hermitean conjugate, and with the help of the symmetries (8) and (23):

$$\begin{aligned} T_{mn}P &= T_{m+1,n} + \frac{\mathrm{i}n}{2}\left[1 + \frac{(1-(-1)^{m+n})\nu R}{2(m+n)} - \frac{(1+(-1)^{m+n})\nu R}{2(m+n+1)}\right] T_{m,n-1} \\ QT_{mn} &= T_{m,n+1} + \frac{\mathrm{i}m}{2}\left[1 + \frac{(1-(-1)^{m+n})\nu R}{2(m+n)} + \frac{(1+(-1)^{m+n})\nu R}{2(m+n+1)}\right] T_{m-1,n} \\ T_{mn}Q &= T_{m,n+1} - \frac{\mathrm{i}m}{2}\left[1 + \frac{(1-(-1)^{m+n})\nu R}{2(m+n)} - \frac{(1+(-1)^{m+n})\nu R}{2(m+n+1)}\right] T_{m-1,n}. \end{aligned} \tag{62}$$

We prove (61) by induction on m. For $m = 0$ the left-hand side reads (using (36))

$$PQ^n = PQQ^{n-1} = -\mathrm{i}\left(N + 1 + \tfrac{1}{2}\nu R\right)Q^{n-1},$$

whereas for the right-hand side we have

$$T_{1n} - \frac{\mathrm{i}n}{2}\left[1 + \frac{(1-(-1)^n)\nu R}{2n} + \frac{(1+(-1)^n)\nu R}{2(n+1)}\right] Q^{n-1}.$$

Upon substituting for T_{1n} from equation (41) we find that the two sides are equal.

If we now assume that (61) is valid for any fixed m, and for all n, then by taking the commutator of each side with P^2 and using (30), we find that (61) is also valid for $m + 1$ for all n, which completes the proof of (61). We have therefore established the $j = 1/2$ case of (54) for all values of j', m, m':

$$\mathcal{P}_{\frac{1}{2}m}\mathcal{P}_{j'm'} = \sum_{k=j'\pm\frac{1}{2}} \Delta\left(\tfrac{1}{2}, j', k, \nu R\right) C^{j'\ \frac{1}{2}\ k}_{m'\ m\ m+m'} \mathcal{P}_{k,m+m'}, \tag{63}$$

and hence may identify the coefficients $\Delta\left(\frac{1}{2}, j, j+\frac{1}{2}, y\right) = \sqrt{2j+1}$ and $\Delta\left(\frac{1}{2}, j, j-\frac{1}{2}, y\right)$ as given in (57).

We now prove the product law by induction on j. We assume that (54) is valid for some fixed j and for all possible values of j', m, m', and show that then (54) is valid also for $j + \frac{1}{2}$. We have already established the $j = 1/2$ case for all values of j', m, m' and hence, setting $j \to \frac{1}{2}$, $j' \to j$, $k \to j + \frac{1}{2}$ in (55), we have

$$\sqrt{2j+1}\mathcal{P}_{j+\frac{1}{2},m'} = \sum_{m_1+m_2=m'} C^{j\ \frac{1}{2}\ j+\frac{1}{2}}_{m_1\ m_2\ m'} \mathcal{P}_{\frac{1}{2},m_2}\mathcal{P}_{j,m_1}.$$

We multiply both sides on the right by $\mathcal{P}_{lm}$ and expand the product $\mathcal{P}_{\frac{1}{2},m_2}\mathcal{P}_{j,m_1}\mathcal{P}_{lm}$ firstly by using the assumed validity of the product law (54) for j to expand $\mathcal{P}_{j,m_1}\mathcal{P}_{lm}$; and secondly by using (63) to expand the product of $\mathcal{P}_{\frac{1}{2},m_2}$ and $\mathcal{P}_{l',m+m_1}$ which occurs in the expansion of $\mathcal{P}_{j,m_1}\mathcal{P}_{lm}$. The result is

$$\begin{aligned} \sqrt{2j+1}\mathcal{P}_{j+\frac{1}{2},m'}\mathcal{P}_{lm} &= \sum_{k,l'} \Delta(j, l, l', -\nu R)\Delta\left(\frac{1}{2}, l', k, \nu R\right)\mathcal{P}_{k,m+m'} \\ &\quad \times \sum_{m_1+m_2=m'} C^{j\ \frac{1}{2}\ j+\frac{1}{2}}_{m_1\ m_2\ m'} C^{l\ j\ l'}_{m\ m_1\ m+m_1} C^{l'\ \frac{1}{2}\ k}_{m+m_1\ m_2\ m+m'}. \end{aligned} \tag{64}$$

The sum over $m_1 + m_2 = m'$ is equal to

$$\sqrt{(2j+2)(2l'+1)}\,W\left(l, j, k, \tfrac{1}{2}; l', j+\tfrac{1}{2}\right)C^{l\ j+\frac{1}{2}\ k}_{m\ m'\ m+m'},$$

where W denotes a Racah coefficient, as follows from the sum given in [24], equation (3.267) (p 108), where we identify $a = l, b = j, c = k, d = \frac{1}{2}, e = l', f = j + \frac{1}{2}$ and

$\alpha = m, \beta = m_1, \gamma = m', \delta = m_2$. This Racah coefficient may itself be summed, using the formula for $W(a, b, c, d; e, b + d)$ in [24], equation (3.288) (p 113).

The sum over l' in equation (64) may now be performed using

$$\sqrt{2j+1}\Delta\left(j+\frac{1}{2}, l, k, \nu R\right) = \sum_{l'=k\pm\frac{1}{2}} \Delta(j, l, l', -\nu R)\Delta\left(\frac{1}{2}, l', k, \nu R\right) \times \sqrt{(2j+2)(2l'+1)}W\left(l, j, k, \frac{1}{2}; l', j+\frac{1}{2}\right), \tag{65}$$

which is equivalent to the defining relations (58) for the coefficients $\Delta(j, k, l, \nu R)$, as follows after substituting for

$$W\left(l, j, k, \frac{1}{2}; k+\frac{1}{2}, j+\frac{1}{2}\right) = \left[\frac{\left(l-j+k+\frac{1}{2}\right)\left(l+j-k+\frac{1}{2}\right)}{(2j+1)(2j+2)(2k+1)(2k+2)}\right]^{\frac{1}{2}},$$

$$W\left(l, j, k, \frac{1}{2}; k-\frac{1}{2}, j+\frac{1}{2}\right) = \left[\frac{(l+j+k+\frac{3}{2})(-l+j+k+\frac{1}{2})}{(2j+1)(2j+2)(2k)(2k+1)}\right]^{\frac{1}{2}}.$$

The result of performing the sum shown in (65) is that the right-hand side of equation (64) reduces to

$$\sqrt{2j+1}\sum_k \Delta\left(j+\frac{1}{2}, l, k, \nu R\right) C^{lj+\frac{1}{2}k}_{mm'm+m'}\mathcal{P}_{k,m+m'}$$

which, when equated to the left-hand side of (64), establishes the case $j + \frac{1}{2}$ of the product law (54). This completes the induction proof.

5.3. Properties of the product law

There are many useful special cases of the product law, for example we may generalize formulae (61), (62) by choosing $m' = j'$ in (54). In this case the Clebsch–Gordan coefficient may be summed to a single term so that we obtain, in terms of T_{mn},

$$T_{mn}Q^r = \sum_{s=0}^{\min(m,r)} \frac{m!r!\mathfrak{P}\left(\frac{m+n}{2}, \frac{r}{2}, -s+\frac{n+m+r}{2}, \nu R\right)}{(2\mathrm{i})^s s!(m-s)!(r-s)!} T_{m-s,n+r-s}. \tag{66}$$

From this formula there follows by symmetry, or by choosing $m' = -j'$, or $m = \pm j$ in (54), similar expressions for each of $Q^r T_{mn}$, $P^r T_{mn}$ and $T_{mn}P^r$ as linear combinations of Weyl-ordered polynomials, with coefficients depending on R through the polynomial $\mathfrak{P}$.

A further special case follows by choosing $n = 0$ in (66), which then expresses $P^m Q^r$ as a linear combination of Weyl-ordered polynomials. It follows that any operator $F(P, Q)$ which can be expanded as a sum over the multinomials $P^m Q^r$ may also be expanded in terms of Weyl-ordered polynomials, where the coefficients could depend on R.

The commutator or anticommutator $[\mathcal{P}_{jm}, \mathcal{P}_{j'm'}]_\pm$ can be evaluated directly from the product law for any (j, m) and (j', m'). A special case comprises the commutators $[T_{mn}, Q^r]$ and $[T_{mn}, P^r]$ which can be evaluated from (66) and its symmetric counterparts.

The product law also implies the existence of further properties of the special functions $F_{mn}(x, y)$ and $\Delta(j, k, l, y)$. By expressing $\mathcal{P}_{jm}$ in the product law (54) in terms of T_{mn} according to definition (25), and then with the help of equation (40) further expressing these polynomials in terms of the functions $F_{mn}(x, y)$ defined by equation (39), we obtain a very general identity relating the polynomials $F_{mn}(x, y)$ and $\Delta(j, k, l, y)$. As a specific example, we may substitute expression (40) for T_{mn} into equation (66) to obtain

$$F_{mn}(x, y) = \sum_{s=0}^{\min(m,r)} \frac{m!r!(-1)^s \mathfrak{P}\left(\frac{m+n}{2}, \frac{r}{2}, -s + \frac{n+m+r}{2}, y\right)}{2^s s!(m-s)!(r-s)!} F_{m-s,n+r-s}(x, y),$$

where on the right-hand side the integer r may take any convenient non-negative value.

The inverted product law (55) expresses $\Delta(j, k, l, y)$ as a sum over the product of two $F_{mn}(x, y)$ functions and may be used to obtain a closed form expression for $\Delta(j, k, l, y)$. If we evaluate (55) for $m'' = k$, then we find the following explicit expression for $\mathfrak{P}(j, k, l, y)$:

$$\mathfrak{P}(j, k, l, y) = \frac{2^{j+k-l}(j+k-l)!(2l+1)!}{(j+k+l+1)!} \sum_{s=0}^{\min(2j, j+k-l)} \frac{(-1)^s F_{s,2j-s}(x, y)}{s!(j+k-l-s)!} \times F_{j+k-l-s,-j+k+l+s}(x - 2j + 2s, y(-1)^{2j}),$$

where the right-hand side is independent of x after the sum is performed, provided that the triangle condition on the triple (j, k, l) is satisfied.

As a consequence of the product law the functions $\Delta(j, k, l, \nu R)$ satisfy a transformation property which follows from the associativity of the operator product:

$$(\mathcal{P}_{jm_1}\mathcal{P}_{km_2})\mathcal{P}_{lm_3} = \mathcal{P}_{jm_1}(\mathcal{P}_{km_2}\mathcal{P}_{lm_3}).$$

By substituting for the product law (54), and using orthogonality and other standard relations of Clebsch–Gordan coefficients and Racah coefficients, we find

$$\Delta(j, k, j', \nu R)\Delta(j', l, l', \nu R) = \sum_{k'} \sqrt{(2k'+1)(2j'+1)} \times \Delta(k, l, k', (-1)^{2j}\nu R)\Delta(j, k', l', \nu R) W(l, k, l', j; k'j'). \tag{67}$$

In terms of the parameters $a = l, b = k, c = l', d = j, e = k', f = j'$, and denoting $y = \nu R$, this reads

$$\Delta(d, b, f, y)\Delta(f, a, c, y) = \sum_{e} \sqrt{(2e+1)(2f+1)} \times \Delta(b, a, e, (-1)^{2d} y)\Delta(d, e, c, y) W(a, b, c, d; e, f),$$

in which form this relation is readily seen to reduce to that for $y = 0$ as derived in [14] (p 250).

Relation (67) generalizes the defining recurrence relation (58) for $\Delta(j, k, l, y)$ which is equivalent to equation (65), as is evident on setting $j = \frac{1}{2}$ and $k = j' - \frac{1}{2}$ in (67). Further properties of the functions $\Delta(j, k, l, y)$ may be derived from (67), for example by choosing $j' = l$ and $l' = 0$, for which the sum reduces to a single term with $k' = j$, we obtain a symmetry relation which in terms of the polynomials $\mathfrak{P}$ reads:

$$\mathfrak{P}(j, k, l, y)\mathfrak{P}(l, l, 0, y) = \mathfrak{P}(l, k, j, y)\mathfrak{P}(j, j, 0, y),$$

where the explicit expression for $\mathfrak{P}(j, j, 0, y)$ is given by (60).

6. Applications of Weyl-ordered polynomials

Having developed in detail the algebraic properties of Weyl-ordered polynomials we now discuss briefly three examples showing how such polynomials can be used to solve quantum mechanical problems. The first example, which originally motivated the consideration of ordering methods, is the construction of suitable quantum mechanical Hamiltonians $H(P, Q)$, and generally quantum mechanical operators $F(P, Q)$. If a classical function $f(p, q)$ can be expanded as an infinite series in the basis of multinomials $p^m q^n$, including possibly negative values of m or n, then the corresponding quantum mechanical operator $F(P, Q)$ is given by the same series expansion but with the basis vectors $p^m q^n$ replaced by $T_{mn}(P, Q)$. We use the property, therefore, that the set $\{T_{mn}(P, Q)\}$ forms a basis in the space of operators in P, Q following the result of the previous section (see equation (66)) which shows that $P^m Q^n$ can be expressed as a linear combination of Weyl-ordered polynomials for any positive integers m, n.

In our formulation of fractional-dimensional quantum mechanics, for which there are three operators P, Q, R satisfying the defining relations (1), operators such as Hamiltonians are constructed from P, Q, R in a suitably ordered form. It is not necessary, however, to specify an ordering with respect to R, since any function of R can be written as a linear combination of R and the identity operator, by anticommuting R to the right of any operators P, Q and using $R^2 = 1$. Hence, Weyl-ordering needs to be performed with respect to P, Q only and so quantum mechanical operators F take the form

$$F(P, Q, R) = \sum_{m,n} T_{mn}(P, Q)(\alpha_{mn} + i^{m+n}\beta_{mn} R) \tag{68}$$

for real or complex coefficients α_{mn}, β_{mn}. If F is Hermitean the coefficients α_{mn}, β_{mn} are all real, as follows from (11) and the equation $RT_{mn} = (-1)^{m+n} T_{mn} R$.

Besides their use in the construction of Hamiltonians, Weyl-ordered polynomials furnish an infinite number of independent unitary operators of the form $\exp(\mathrm{i}\lambda T_{mn})$ (for $\lambda \in \mathbb{R}$) which may be used to perform quantum canonical transformations and to simplify Hamiltonian expressions by means of unitary transformations. An example is $U = \exp(\mathrm{i}\lambda Q^2)$ used in [5] to solve the time-dependent harmonic oscillator in fractional dimensions; see also [25, 26] for applications to one-dimensional models which can also be generalized to fractional dimensions.

6.1. Time evolution of quantum operators

As a second, more specific, example of the use of Weyl-ordered polynomials we consider the time evolution of the operators P, Q, R, following the discussion by Bender and Dunne [12, 27] for the case $\nu = 0$. The quantum equations of motion for a Hamiltonian $H(P, Q, R)$ are

$$\dot{Q} = \mathrm{i}[H, Q], \qquad \dot{P} = \mathrm{i}[H, P], \qquad \dot{R} = \mathrm{i}[H, R]$$

which in general comprises a set of nonlinear ordinary differential equations for the unknowns, the noncommuting operators P, Q, R. Assuming that R commutes with H, we need to solve for $P(t)$, $Q(t)$ only, and to this end we look for an (Hermitean) operator $F(P, Q)$ such that

$$\mathrm{i}[H, F] = 1.$$

In effect we perform a quantum canonical transformation, as discussed in [3]. Given such an operator F we may integrate $\dot{F} = 1$ directly to obtain

$$F(P(t), Q(t)) = t + F(P_0, Q_0)$$

where $P_0 = P(0)$, $Q_0 = Q(0)$. We regard this, together with

$$H(P(t), Q(t)) = H(P_0, Q_0),$$

as an implicit solution to the equations of motion. The operator F may be expanded as shown in equation (68), and the relation $\mathrm{i}[H, F] = 1$ is then satisfied provided that the coefficients

α_{mn}, β_{mn}, generally infinite in number, satisfy partial difference equations. These equations may be converted to first order linear partial differential equations, see [12] for a discussion and further details.

A simple example of this procedure is the free particle Hamiltonian $H = \frac{1}{2}P^2$, for which we choose

$$F(P, Q) = \tfrac{1}{2}(P^{-1}Q + QP^{-1}) = T_{-1,1}(P, Q),$$

which satisfies $\mathrm{i}[H, F] = 1$ (for any ν). The integrated equations are

$$P(t)^{-1}Q(t) + Q(t)P(t)^{-1} = P_0^{-1}Q_0 + Q_0P_0^{-1} + 2t, \qquad P(t)^2 = P_0^2,$$

which leads to the solution

$$P(t) = P_0, \qquad Q(t) = Q_0 + tP_0.$$

Many of the models discussed in [12] have properties which generalize to fractional dimensions, including the case where $F(P, Q)$ is a function of either P or Q only. As an example consider the Hamiltonian $H = T_{1n}(P, Q)$ where n is odd, then from the commutation relation

$$[Q^r, T_{1n}] = \mathrm{i}rQ^{n+r-1}, \qquad (n \text{ odd})$$

which is valid for integers r (positive or negative), we determine that

$$F = \frac{Q^{1-n}}{1-n}$$

satisfies $\mathrm{i}[H, F] = 1$. The equation $\dot{F} = 1$ is equivalent to $\dot{Q} = Q^n$ and is readily solved. For $n = -1$ the solution is related to that of the free particle discussed above.

For the harmonic oscillator $H = \frac{1}{2}(P^2 + Q^2)$ we can represent F as an infinite series:

$$F(P, Q) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} T_{-2m+1,2m+1}(P, Q), \tag{69}$$

which is the Weyl-symmetrized quantum operator corresponding to the classical angle variable $\theta = \arctan\left(\frac{q}{p}\right)$. As observed in [12], another (unsymmetrized) choice is

$$\widetilde{F}(P, Q) = \arctan(Q(t)P(t)^{-1})$$

which also satisfies $\mathrm{i}[H, \widetilde{F}] = 1$ and also reduces to the classical angle variable. F and $\widetilde{F}$ differ by a function of the Hamiltonian. These observations apply also to fractional dimensions, essentially because relations (22) are true for any ν.

As discussed by Bender and Dunne [12], the convergence and existence of series expansions such as (69), and (68) in general, requires scrutiny for each particular case, considering that for any Hermitean F satisfying $\mathrm{i}[H, F] = 1$ the corresponding unitary operator $U = \exp(\mathrm{i}\lambda F)$ behaves as an energy raising operator, that is $[H, U] = \lambda U$. If U exists for some $\lambda \in \mathbb{R}$ as an operator in the Hilbert space $\mathfrak{H}$, discussed in section 2, then to every eigenfunction ψ of H with eigenvalue E there corresponds another eigenfunction $U\psi$ with eigenvalue $E+\lambda$. For the harmonic oscillator this implies that U does not exist in $\mathfrak{H}$ except for integer values of λ and, for some models, will not exist for any λ even if F itself exists as an operator in $\mathfrak{H}$. An alternative approach is to avoid the infinite series by constructing the raising operator U directly as a finite linear combination of Weyl-ordered polynomials where λ is to be chosen, if possible, such that the infinite series expansion in (68) terminates whilst satisfying $[H, U] = \lambda U$. For the harmonic oscillator the coefficients α_{mn} can be chosen such that $\lambda = \pm 1$ leading to $U = Q \pm \mathrm{i}P$, and the time evolution of the model is determined by solving $\dot{U} = \mathrm{i}U$.

6.2. Time-dependent Hamiltonians

As a third example of the application of Weyl-ordered polynomials we consider the time-dependent Schrödinger equation

$$\left(\mathrm{i}\frac{\partial}{\partial t} - H\right)\psi(x,t) = 0,$$

where $H = H(t)$ depends explicitly on time. In the method of Lewis and Riesenfeld [28, 29] one looks for a nontrivial Hermitean operator $I(t)$ which satisfies

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \mathrm{i}[H, I] = 0, \tag{70}$$

from which it follows that the eigenvalues λ of $I(t)$ are time independent. After determining the eigenfunctions $\phi_\lambda(x,t)$ of $I(t)$ one constructs solutions of the Schrödinger equation in the form $\psi_\lambda(x,t) = \mathrm{e}^{\mathrm{i}\alpha_\lambda(t)}\phi_\lambda(x,t)$ where the phase $\alpha_\lambda(t)$ is given by

$$\frac{\mathrm{d}\alpha_\lambda}{\mathrm{d}t} = \left(\phi_\lambda, \left(\mathrm{i}\frac{\partial}{\partial t} - H\right)\phi_\lambda\right).$$

It remains therefore to solve (70), which can be achieved in principle by expanding $I(t)$ in terms of Weyl-ordered polynomials as shown in equation (68), where the coefficients $\{\alpha_{mn}, \beta_{mn}\}$ now depend on t and where R is time independent, assuming that $[H, R] = 0$. The commutator $[H, I]$ can also be expressed as a linear combination of Weyl-ordered polynomials using the product law and hence, equating both sides, we obtain a possibly infinite set of ordinary differential equations for $\{\alpha_{mn}(t), \beta_{mn}(t)\}$. These equations can be converted to partial differential equations and subsequently solved (see [30]) in a way similar to the partial difference equations occurring in the papers of Bender and Dunne [12].

Of particular significance are models for which H, and subsequently also $I(t)$, can be expressed as a linear combination of a finite number of elements T_{mn} which are closed under commutation and which therefore form a dynamical Lie algebra, as discussed for example in [31–33]. The set $\{T_{20}, T_{11}, T_{02}\}$, which generates $\mathfrak{sl}_2(\mathbb{C})$, is a particular example which has been used by Lewis and Riesenfeld [28, 29] in dimension $d = 1$ to solve the time-dependent harmonic oscillator, and has been generalized to any fractional dimension d in [5] (equation (41)). Various authors have investigated models such as Ermakov systems by using a generalization (for $d = 1$) of the realization (24), see for example [25, 26, 34, 35] where the $\mathfrak{sl}_2(\mathbb{C})$ generators are given for any ν by

$$\begin{aligned}
J_0 &= \frac{\mathrm{i}}{4}(PQ + QP) + \alpha \\
J_+ &= -\frac{1}{2}Q^2 \\
J_- &= -\frac{1}{2}P^2 + \mathrm{i}\alpha(PQ^{-1} + Q^{-1}P) + \beta Q^{-2},
\end{aligned}$$

where α, β are time-independent constants. Whilst this realization generalizes to fractional dimensions the same is not true of all dynamical algebras. Recently there has been discussion [36, 37] of the Schrödinger equation for $d = 1$ for time-dependent linear potentials, in which the Hamiltonian H and the invariant I are each expressed as linear combinations of the six operators $\{T_{00}, T_{10}, T_{01}, T_{20}, T_{11}, T_{02}\}$ which form a Lie algebra corresponding to the inhomogeneous symplectic group $\mathrm{ISp}(2, \mathbb{R})$ [31]. The time-dependent coefficients in the expansion of $I(t)$ appear as the solutions of three first order coupled ordinary differential equations which are readily solved, however this approach is restricted to dimension $d = 1$ since for nonzero ν the algebra of these six operators is infinite dimensional.

Acknowledgment

We thank the referees for some very helpful suggestions.

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